Journal of Statistical Physics, Vol. 84, Nos. 1/2, 1996

Ising Models on the Lattice Sierpinski Gasket

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Received October 31, 1995

Ferromagnetic Ising models on the lattice Sierpinski gasket are considered. We prove the Dobrushin–Shlosmann mixing condition and discuss corresponding properties of the stochastic Ising models.

KEY WORDS: Ising model; lattice Sierpinski gasket; Dobrushin–Shlosmann mixing condition.

1. INTRODUCTION

This work originates in an attempt to understand more about the enormous difference between one-dimensional and two-dimensional Ising models. Our interest in the lattice Sierpinski gasket comes from its geometrical character as a cross between the one-dimensional and the two-dimensional lattice. The lattice Sierpinski gasket is locally two dimensional, since each point in the lattice has four nearest neighbors. However, it is in a sense closer to the one-dimensional lattice. For example, there exists a sequence $\{W_l\}_{l=1}^{\infty}$ of finite sets which increases to the whole lattice keeping $|\partial W_l| = 8$. It is therefore reasonable to expect that a fairly strong mixing property survives for an Ising model on the lattice Sierpinski gasket even at low temperature. The purpose of this article is to vindicate this expectation from both the static and dynamical points of view. Before we can state the results precisely, we have to introduce a long sequence of definitions and notations.

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0022-4715/96/0700-0295$09.50/0

The Lattice Sierpinski Gasket. We will denote the usual Euclidean distance on $\mathbf{R}^2$ by $d(x, y) = |x - y|$. We define $x^* = (-x_1, x_2)$ for $x = (x_1, x_2) \in \mathbf{R}^2$ and $A^* = \{x^*; x \in A\}$ for $A \subset \mathbf{R}^2$. Set $x_l = 2^l(1, 0)$ and $y_l = 2^l(1/2, \sqrt{3}/2)$ $(l = 0, 1,...)$. Starting with $A_0 = \{0, x_0, y_0\}$, we construct a sequence $(A_l)_{l=0}^{\infty}$ of finite subsets in $\mathbf{R}^2$ inductively by

$$A_{l+1} = \bigcup_{\nu=0}^{2} A_{l,\nu}$$

where

$$A_{l,\nu} = \begin{cases} A_l & \text{if} \quad \nu = 0 \\ A_l + x_l & \text{if} \quad \nu = 1 \\ A_l + y_l & \text{if} \quad \nu = 2 \end{cases}$$

Now, we set $W_l = A_l^* \cup A_l$ and define the lattice Sierpinski gasket G by

$$G = \bigcup_{l=0}^{\infty} W_l$$

The number of points contained in $A \subset G$ will be denoted by $|A|$ and we write $A \subset\subset G$ when $A \subset G$ and $1 \leqslant |A| < \infty$. A set $W \subset\subset G$ is said to be an *l-pair* with the joint $q \in G$ if there exists $\{x, x'\} \subset G$ such that

$$W = (A_l + x) \cup (A_l + x')$$

$$\{q\} = (A_l + x) \cap (A_l + x')$$

For example, W_l is an l-pair with the origin as its joint.

The set $\mathbf{B}$ of *bonds* in G is a subset of $\{\{x, y\} \subset G;\ |x - y| = 1\}$ defined as follows. Let $\mathbf{B}_0 = \{b \subset A_0;\ |b| = 2\}$ and

$$\mathbf{B}_{l+1} = \bigcup_{\nu=0}^{2} \mathbf{B}_{l,\nu}, \qquad l = 0, 1,...$$

where

$$\mathbf{B}_{l,\nu} = \begin{cases} \mathbf{B}_l & \text{if} \quad \nu = 0 \\ \{b + x_l;\ b \in \mathbf{B}_l\} & \text{if} \quad \nu = 1 \\ \{b + y_l;\ b \in \mathbf{B}_l\} & \text{if} \quad \nu = 2 \end{cases}$$

Finally, define **B** by

$$\mathbf{B} = \bigcup_{n=0}^{\infty} \mathbf{B}_n^* \cup \mathbf{B}_n$$

where $\mathbf{B}_n^* = \{b^*: b \in \mathbf{B}_n\}$. For a set $\Lambda \subset G$, we also define

$$\mathbf{B}_\Lambda = \{\{x, y\} \in \mathbf{B}; (x, y) \in \Lambda^2\}$$
$$\partial\Lambda = \{\{x, y\} \in \mathbf{B}; x \in \Lambda, y \notin \Lambda\}$$
$$\partial_{\text{int}}\Lambda = \{x \in \Lambda; x \in b \text{ for some } b \in \partial\Lambda\}$$
$$\partial_{\text{ext}}\Lambda = \{x \notin \Lambda; x \in b \text{ for some } b \in \partial\Lambda\}$$

The Configuration. The set of all spin configurations of $S = \{-1, 1\}$ on $\Lambda \subset G$ is denoted by S^Λ,

$$S^\Lambda = \{\sigma = (\sigma_x)_{x \in \Lambda}; \sigma_x \in S\}, \qquad \Lambda \subset G$$

As usual, S^Λ is endowed with the product topology inherited from the discrete topology on S. In S^Λ, the following partial order is introduced:

$$\sigma \leqslant \sigma' \qquad \text{if} \quad \sigma_x \leqslant \sigma'_x \quad \text{for all } x \in \Lambda$$

Clearly, the maximal and the minimum element in this partial order are $+\mathbf{1}$ and $-\mathbf{1}$, which are respectively the configurations with all spins $+1$ and -1. For $f; S^\Lambda \to \mathbf{R}$ and $x \in \Lambda$, we set $\nabla_x f(\sigma) = f(\sigma^x) - f(\sigma)$, where $\sigma^x = (\sigma_y^x)_{y \in \Lambda}$ is the confuguration obtained from σ by flipping the spin at x,

$$\sigma_y^x = \begin{cases} -\sigma_x & \text{if } y = x \\ \sigma_y & \text{if } y \neq x \end{cases}$$

For $S^\Lambda \to \mathbf{R}$, we introduce the notations

$$\Delta_f = \{x \in \Lambda; \nabla_x f \text{ is not identically zero}\}$$
$$\|f\| = \sup_{\sigma \in S^\Lambda} |f(\sigma)|$$
$$|||f||| = \sum_{x \in \Lambda} \|\nabla_x f\|$$

Function spaces $\mathscr{C}$ and $\mathscr{C}_\Lambda$ ($\Lambda \subset G$) are defined respectively by

$$\mathscr{C} = \{f: S^G \to \mathbf{R}; |\Delta_f| < \infty\}$$
$$\mathscr{C}_\Lambda = \{f: S^G \to \mathbf{R}; \Delta_f \subset \Lambda\}$$

The Local Specification. For a measure m on some measurable space, we wil use the following abbreviations

$$mf = \int f\,dm$$

$$m(f; g) = m(fg) - mf \cdot mg$$

whenever the integrations make sense. Fix $\mathbf{J} = (J_b)_b \in \mathbf{R}^{\mathbf{B}}$ and $\mathbf{h} = (h_x)_x \in \mathbf{R}^G$. We will refer to $\mathbf{J}$, $\mathbf{h}$, and the pair $(\mathbf{J}, \mathbf{h})$ respectively as the *coupling constants*, the *magnetic field*, and the *interaction*. For $\Lambda \subset\subset G$ and $\omega \in S^G$, the *Hamiltonian* $H_{\Lambda,\omega} \in \mathscr{C}_\Lambda$ is defined by

$$-H_{\Lambda,\omega}(\sigma) = \sum_{\{x,y\} \in \mathbf{B}_\Lambda} J_{\{x,y\}} \sigma_x \sigma_y + \sum_{x \in \Lambda} \sigma_x \left(h_x + \sum_{y:\{x,y\} \in \partial\Lambda} J_{\{x,y\}}\, \omega_y \right)$$

The $(\mathbf{J}, \mathbf{h})$-*local specification* is a family $\{\mu^{\Lambda,\omega};\ \Lambda \subset\subset G,\ \omega \in S^G\}$ of Borel probability measures on Σ^G defined by

$$\mu^{\Lambda,\omega} f = \delta_\omega \left(\frac{\rho^\Lambda (f \exp - H_{\Lambda,\omega})}{\rho^\Lambda \exp - H_{\Lambda,\omega}} \right)$$

where ρ^Λ is the 1/2-Bernoulli measure on Σ^Λ and δ_ω is the Dirac delta measure concentrated on the configuration ω.

The Infinite-Volume Gibbs State. A Borel probability measure μ is called a $(\mathbf{J}, \mathbf{h})$-*infinite-volume Gibbs state* or simply a $(\mathbf{J}, \mathbf{h})$-*Gibbs state* if it solves the following Dobrushin–Lanford–Ruelle equation:

$$\mu(\mu^{\Lambda,\cdot} f) = \mu f, \qquad \forall \Lambda \subset\subset G, \qquad \forall f \in \mathscr{C} \tag{1.1}$$

By a slight extension of the arguments in ref. 2, p. 356, we see that if $\sup_{b \in \mathbf{B}} |J_b| < \infty$, then there exists a unique $(\mathbf{J}, \mathbf{h})$-Gibbs state μ on S^G. This in turn implies that

$$\lim_{\Lambda \uparrow G} \sup_{\omega \in S^G} |\mu^{\Lambda,\omega} f - \mu f| = 0 \qquad \text{for all } \ f \in \mathscr{C} \tag{1.2}$$

[see ref. 5, p. 120, (7.10), for example].

The Stochastic Dynamics. We introduce now for the above Ising model the time evolution called the stochastic Ising model or Glauber dynamics. To do this, we now suppose there exist $J \geqslant 0$ and $h \geqslant 0$ such that $\mathbf{J} \in [-J, J]^{\mathbf{B}}$ and $\mathbf{h} \in [-h, h]^G$. We define for each $x \in G$ an operator $A_x: \mathscr{C} \to \mathscr{C}$ by $A_x f = c_2 \nabla_x f$, where the flip rate $c_x: S^G \to (0, \infty)$ is a function on which we assume the following:

(R-1) Boundedness: There exist positive constants $\underline{c}(J, h)$ and $\bar{c}(J, h)$ such that

$$\underline{c}(J, h) \leqslant c_x(\sigma) \leqslant \bar{c}(J, h) \qquad \text{for all} \quad (x, \sigma) \in G \times S^G$$

(R-2) Finite range: There exists $r \geqslant 0$ such that $\nabla_y c_x \equiv 0$ if $|x-y| > r$.

(R-3) Detailed balanced condition:

$$\nabla_x \{ c_x(\sigma) \exp - H_{\{x\},\sigma}(\sigma) \} \equiv 0 \qquad \text{for all} \quad x \in G \tag{1.3}$$

We introduce the stochastic Ising model in a finite set $\Lambda \subset\subset G$ with the boundary condition $\omega \in S^G$. We define an operator $A^{\Lambda,\omega}$: $\mathscr{C} \to \mathscr{C}$ by

$$A^{\Lambda,\omega} f(\sigma) = \sum_{x \in \Lambda} A_x f(\sigma|_\Lambda \cdot \omega|_{\Lambda^c}), \qquad f \in \mathscr{C}$$

where $\sigma|_\Lambda \cdot \omega|_{\Lambda^c}$ denotes the following configuration:

$$(\sigma|_\Lambda \cdot \omega|_{\Lambda^c})_x = \begin{cases} \sigma_x & \text{if} \quad x \in \Lambda \\ \omega_x & \text{if} \quad x \notin \Lambda \end{cases}$$

We then have

$$-\mu^{\Lambda,\omega}(f A^{\Lambda,\omega} g) = \tfrac{1}{2} \sum_{x \in \Lambda} \mu^{\Lambda,\omega}(c_x \nabla_x f \nabla_x g), \qquad \{f, g\} \subset \mathscr{C} \tag{1.4}$$

We set

$$T_t^{\Lambda,\omega} = \exp t A^{\Lambda,\omega}, \qquad t > 0$$

$A^{\Lambda,\omega}$ and $T_t^{\Lambda,\omega}$ can also be regarded as operators from $\mathscr{C}_\Lambda$ into itself.

2. RESULTS

We will show the following property for the local specification, which is called complete analyticity or the Dobrushin–Shlosmann mixing condition.

Theorem 2.1. For any $J \geqslant 0$, there exists $C_0 \in (0, \infty)$ such that the following holds for any $\mathbf{J} \in [0, J]^{\mathbf{B}}$, $\mathbf{h} \in \mathbf{R}^G$, $\Lambda \subset\subset G$, $y \notin \Lambda$, and $f \in \mathscr{C}_\Lambda$:

$$\sup_{\omega \in S^G} |\mu^{\Lambda,\omega^y} f - \mu^{\Lambda,\omega} f| \leqslant C_0 \, |||f||| \exp - \frac{d(\Delta_f, y)}{C_0} \tag{2.1}$$

Starting from Theorem 2.1, we can derive ergodic properties of the stochastic Ising model.

Theorem 2.2. For any $J \geqslant 0$ and $h \geqslant 0$, there exists $\{C_1, C_2, C_3\} \subset (0, \infty)$ such that the following hold for any $\mathbf{J} \in [0, J]^{\mathbf{B}}$, $\mathbf{h} \in [-h, h]^G$, $\Lambda \subset\subset G$, and $f \in \mathscr{C}_\Lambda$:

$$\text{(i)} \qquad \mu^{\Lambda,\omega}(|f-\mu^{\Lambda,\omega}f|^2) \leqslant C_1 \sum_{x \in \Lambda} \mu^{\Lambda,\omega}(|\nabla_x f|^2) \tag{2.2}$$

$$\text{(ii)} \qquad \mu^{\Lambda,\omega}\left(f^2 \log \frac{f^2}{\mu^{\Lambda,\omega}(f^2)}\right) \leqslant C_2 \sum_{x \in \Lambda} \mu^{\Lambda,\omega}(|\nabla_x f|^2) \tag{2.3}$$

$$\text{(iii)} \qquad \|T_t^{\Lambda,\omega} f - \mu^{\Lambda,\omega} f\| \leqslant C(f) \exp -\frac{t}{C_3} \tag{2.4}$$

where $C(f) > 0$ is a function of $f \in \mathscr{C}$, which does not depend on the choice of Λ and ω.

Proof. The equivalence of the conditions (2.1)–(2.4) is well established in the cubic lattice case (e.g., refs. 13 and 10) and almost the same proof works in our setting. So we will just indicate the references. It is well known that (ii) implies (i) [ref. 3, p. 224, (6.1.7)]. Using Theorem 2.1, we can prove (ii) just in the same way as ref. 10, Theorem 3. With (ii) in hand, we can follow the arguments in ref. 12, Lemma 2.9 to conclude (iii). QED

3. PROOF OF THEOREM 2.1

In this section, we will prove Theorem 2.1. We assume that the coupling constants $\mathbf{J}$ satisfy $\mathbf{J} \in [0, J]^{\mathbf{B}}$ for some $J \geqslant 0$. For the moment, we will consider for each $\Lambda \subset\subset G$ a magnetic field $\mathbf{h} = (h_x^\Lambda)_{x \in \Lambda} \in \mathbf{R}^\Lambda$, which may be Λ-dependent. We define a probability measure $\mu^{\Lambda,\mathbf{h}}$ on Σ^Λ by

$$\mu^{\Lambda,\mathbf{h}} f = \frac{\rho^\Lambda(f \exp - H_{\Lambda,\mathbf{h}})}{\rho^\Lambda \exp - H_{\Lambda,\mathbf{h}}}, \qquad f \in \mathscr{C}_\Lambda \tag{3.1}$$

where

$$-H_{\Lambda,\mathbf{h}}(\sigma) = \sum_{\{x,y\} \in \mathbf{B}_\Lambda} J_{\{x,y\}} \sigma_x \sigma_y + \sum_{x \in \Lambda} h_x^\Lambda \sigma_x \tag{3.2}$$

When $h_x^\Lambda \equiv 0$, we will omit the index $\mathbf{h}$:

$$\mu^\Lambda = \mu^{\Lambda,\mathbf{h}}|_{h_x^\Lambda \equiv 0}$$

$$H_\Lambda = H_{\Lambda,\mathbf{h}}|_{h_x^\Lambda \equiv 0}$$

The measure $\mu^{\Lambda,\mathbf{h}}$ defined above will be called the *finite-volume Gibbs state* with respect to the Hamiltonian (3.2), to which we can apply standard correlation inequalities such as FKG, GHS, GKS1, and GKS2.[4] Also, the following inequality due to E. Lieb is useful for our purpose.

Lemma 3.1.[8] Suppose that $X \subset\subset G$, $Y \subset\subset G$, $h_x^X \equiv 0$, and $h_x^Y \equiv 0$. Then, for any $(x, y) \in X \backslash Y \times Y \backslash X$,

$$\mu^{H_X + H_Y}(\sigma_x \sigma_y) \leqslant \sum_{z \in X \cap Y} \mu^X(\sigma_x \sigma_z)\, \mu^{H_X + H_Y}(\sigma_z \sigma_y) \tag{3.3}$$

where

$$\mu^{H_X + H_Y} f = \frac{\rho^{X \cup Y}(f \exp(-H_X - H_Y))}{\rho^{X \cup Y} \exp(-H_X - H_Y)}$$

Proof. This is (20) of ref. 8. QED

Although G is not a homogeneous object like $\mathbf{Z}^d$, the nonhomogeneity can be circumvented by the following simple observation.

Lemma 3.2. For each $x \in G$ and $l \in \mathbf{N}$, there exist l-pair $U_{l,x}$, $(l+1)$-pair $V_{l+1,x}$, and a point $q_{l,x} \in G$ such that the following hold:

(i) $x \in U_{l,x}$.

(ii) $U_{l,x}$ and $V_{l+1,x}$ have $q_{l,x}$ as the common joint.

Proof. For any $l \in \mathbf{N}$, there exists $X_l \subset G$ such that $G = \bigcup_{x \in X_l} (x + \Lambda_l)$. Let us call each $x + \Lambda_l$ $(x \in X_l)$ an *l-cell.* For each $(l+1)$-cell Λ, there exist another three $(l+1)$-cells $\{\Lambda^{(j)}\}_{j=1}^{3}$ such that $\{\Lambda \cup \Lambda^{(j)}\}_{j=1}^{3}$ are $(l+1)$-pairs. This implies that for any $x \in \Lambda$, there exists a triple $\{U_{l,x}, V_{l+1,x}, q_{l,x}\}$, with the desired properties. QED

Lemma 3.3. For any $J \geqslant 0$, there exists $C_4 \in (0, \infty)$ such that the following holds for any $\Lambda \subset\subset G$, $(J_b)_b \in [0, J]^{\mathbf{B}}$, and $(h_x^\Lambda)_x \in [0, \infty)^\Lambda \cup (-\infty, 0]^\Lambda$:

$$0 \leqslant \mu^{\Lambda,\mathbf{h}}(\sigma_x; \sigma_y) \leqslant C_4 \exp -\frac{|x-y|}{C_4} \qquad \text{for all} \quad \{x, y\} \subset \Lambda \tag{3.4}$$

Proof. The left-hand-side inequality is nothing but the FKG inequality. To prove the right-hand-side inequality, it is sufficient to consider the following special case:

$$J_b \equiv J, \qquad \forall b \in \mathbf{B} \tag{3.5}$$

$$h_x^\Lambda \equiv 0, \qquad \forall \Lambda \subset\subset G, \quad \forall x \in \Lambda \tag{3.6}$$

In fact, $\mu^{\Lambda,\mathbf{h}}(\sigma_x; \sigma_y)$ is an even function of $(h_x^\Lambda)_{x\in\Lambda} \in \mathbf{R}^\Lambda$, and is nonincreasing in $[0, \infty)^\Lambda$ by the GHS inequality. When $\mathbf{h} \in [0, \infty)^\Lambda$, $\mu^{\Lambda,\mathbf{h}}(\sigma_x; \sigma_y)$ is also a nondecreasing function of $\mathbf{J}$ by the GKS2 inequality. Therefore, if $(h_x^\Lambda)_{x\in\Lambda} \in [0, \infty)^\Lambda \cup (-\infty, 0]^\Lambda$, then

$$\mu^{\Lambda,\mathbf{h}}(\sigma_x; \sigma_y) \leqslant \mu^{\Lambda,\mathbf{h}}(\sigma_x \sigma_y)|_{h_x^\Lambda \equiv 0, J_b \equiv J}$$

In view of this, we will assume (3.5) and (3.6). We will divide the rest of the proof into three steps, in which we will abbreviate $\mu^{\Lambda,\mathbf{h}}(\sigma_x \sigma_y)$ by $\mu^\Lambda(x, y)$ and $(x_l + y_l)/2$ by z_l.

Step 1. We first show that

$$\alpha_l := \mu^{\Lambda_l}(0, x_l) \xrightarrow{l \to \infty} 0 \tag{3.7}$$

To see this, we consider the unique infinite-volume Gibbs state μ for $J_b \equiv J$ and $h_x \equiv 0$. Then, (1.2) implies that

$$\lim_{l \to \infty} \mu(0, x_l) = 0$$

while (1.2), FKG, and GKS2 yield

$$\sup_{\Lambda \subset\subset G} \mu^\Lambda(0, x_l) = \mu(0, x_l)$$

Combining these two, we obtain (3.7).

Step 2. Define $\delta_l \in [0, 1]$ by

$$\delta_l = \max\{\mu^{W_{l+1}}(z, z'); z \in W_l, z' \in \partial_{\text{int}} W_{l+1}\} \tag{3.8}$$

We next prove that

$$\delta_l \xrightarrow{l \to \infty} 0 \tag{3.9}$$

Set $Y_l = \{x \in W_{l+1};\ |x| \geqslant 2^l\}$. We then have $H_{W_{l+1}} = H_{W_l} + H_{Y_l}$. Therefore, for $z \in W_l \backslash \partial_{\text{int}} W_l$ and $z' \in \partial_{\text{int}} W_{l+1}$, we have by Lieb's inequality (3.3)

$$\mu^{W_{l+1}}(z, z') \leqslant \sum_{w \in \partial_{\text{int}} W_l} \mu^{W_l}(z, w) \mu^{W_{l+1}}(w, z')$$

and thus

$$\delta_l \leqslant 4 \max\{\mu^{W_{l+1}}(w, z');\ w \in \partial_{\text{int}} W_l,\ z' \in \partial_{\text{int}} W_{l+1}\} \tag{3.10}$$

We now estimate the right-hand side of (3.10). Because of the symmetry which comes from the assumptions (3.5) and (3.6), we have only to consider the following three correlations:

$$\mu^{W_{l+1}}(x_l, x_{l+1}), \quad \mu^{W_{l+1}}(x_l, y_{l+1}), \quad \mu^{W_{l+1}}(x_l, x^*_{l+1})$$

For $\mu^{W_{l+1}}(x_l, x_{l+1})$, we have by Lieb's inequality that

$$\begin{aligned}
&\mu^{W_{l+1}}(x_l, x_{l+1}) \\
&\quad \leqslant \mu^{A_{l-1} + (x_l + x_{l+1})/2}\left(x_{l+1}, \frac{x_l + x_{l+1}}{2}\right) \mu^{W_{l+1}}\left(\frac{x_l + x_{l+1}}{2}, x_l\right) \\
&\qquad + \mu^{A_{l-1} + (x_l + x_{l+1})/2}\left(x_{l+1}, \frac{x_{l+1} + z_{l+1}}{2}\right) \mu^{W_{l+1}}\left(\frac{x_{l+1} + z_{l+1}}{2}, x_l\right) \\
&\quad \leqslant 2\alpha_{l-1}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu^{W_{l+1}}(x_l, y_{l+1}) &\leqslant \mu^{A_{l,2}}(y_{l+1}, y_l)\, \mu^{W_{l+1}}(y_l, x_l) \\
&\quad + \mu^{A_{l,2}}(y_{l+1}, z_{l+1})\, \mu^{W_{l+1}}(z_{l+1}, x_l) \\
&\leqslant 2\alpha_l
\end{aligned}$$

and

$$\begin{aligned}
\mu^{W_{l+1}}(x_l, x^*_{l+1}) &\leqslant \mu^{A^*_{l+1}}(x^*_{l+1}, 0)\, \mu^{W_{l+1}}(0, x_l) \\
&\leqslant \alpha_{l+1}
\end{aligned}$$

Therefore, (3.7) implies (3.9).

Step 3. We are now in a position to conclude (3.4). The following argument will be reminiscent of the proof of ref. 11, Theorem 1.2. Take $l \in \mathbf{N}$ satisfying

$$4\delta_l < \exp -1 \tag{3.11}$$

We may assume that $|x-y|>2^l$ and hence that $n2^l<|x-y|\leqslant(n+1)2^l$ for some $n=1, 2,...$. We may also assume that $\Lambda\supset\{z\in G;\ |x-z|\leqslant(n+1)2^l\}$. For each $z\in\Lambda$, take a triple $\{U_{l,z}, V_{l+1,z}, q_{l,z}\}$ in Lemma 3.2; then we obtain by Lieb's inequality and (3.11) that

$$\begin{aligned}
\mu^{\Lambda}(x, y) &\leqslant \sum_{v_1\in\partial_{\mathrm{int}}V_{l+1,x}} \mu^{V_{l+1,x}}(x, v_1)\,\mu^{\Lambda}(v_1, x)\\
&\leqslant \sum_{v_1\in\partial_{\mathrm{int}}V_{l+1,v_1}} \cdots \sum_{v_n\in\partial_{\mathrm{int}}V_{l+1,v_n}} \mu^{V_{l+1,x}}(x, v_1)\\
&\qquad\times\cdots\mu^{V_{l+1,v_{n-1}}}(v_{n-1}, v_n)\,\mu^{\Lambda}(v_n, y)\\
&\leqslant(4\delta_l)^n\\
&\leqslant\exp -n\\
&\leqslant\exp\left(1-\frac{|x-y|}{2^l}\right)
\end{aligned}$$

which proves (3.4). QED

Proof of Theorem 2.1.

Step 1. We will prove that

$$0\leqslant\mu^{\Lambda,\mathbf{h}}(\sigma_x;\sigma_y)\leqslant C_4\exp-\frac{|x-y|}{C_4} \tag{3.12}$$

for all $\Lambda\subset\subset G$ and $\mathbf{h}\in\mathbf{R}^{\Lambda}$, where C_4 is the constant which appears in (3.4). The left-hand-side inequality is nothing but the FKG. To show the right-hand-side inequality, define a bijection

$$\begin{aligned}
&(\sigma^1, \sigma^2)\in(S^{\Lambda})^2\\
&\qquad\mapsto(\sigma^+, \sigma^-)\in T:=\{(\sigma^+, \sigma^-)\in\{0, \pm1\}^{\Lambda};\ (\sigma_x^+)^2+(\sigma_x^-)^2=1\}
\end{aligned}$$

by $\sigma_x^{\pm}=(\sigma_x^1\pm\sigma_x^2)/2$. We then have that

$$\begin{aligned}
&-H_{\Lambda,\mathbf{h}}(\sigma^1)-H_{\Lambda,\mathbf{h}}(\sigma^2)\\
&\quad=\sum_{\{x,y\}\in\mathbf{B}_\Lambda} J_{\{x,y\}}(\sigma_x^1\sigma_y^1+\sigma_x^2\sigma_y^2)+\sum_{x\in\Lambda}(h_x^1\sigma_x^1+h_x^1\sigma_x^2)\\
&\quad=\sum_{\{x,y\}\in\mathbf{B}_\Lambda} 2J_{\{x,y\}}(\sigma_x^+\sigma_y^++\sigma_x^-\sigma_y^-)+\sum_{x\in\Lambda}2h_x^+\sigma_x^+\\
&\quad=-2H_{\Lambda,\mathbf{h}}(\sigma^+)-2H_{\Lambda,0}(\sigma^-)
\end{aligned} \tag{3.13}$$

and thus, for a function $f = f(\sigma^1, \sigma^2)$ on $(S^\Lambda)^2$,

$$(\mu^{\Lambda,\mathbf{J},\mathbf{h}} \otimes \mu^{\Lambda,\mathbf{J},\mathbf{h}}) f = \frac{\sum_{\sigma^1,\sigma^2} f \exp -2\{H_{\Lambda,\mathbf{h}}(\sigma^+) + H_{\Lambda,0}(\sigma^-)\}}{(\hat{Z}^{\Lambda,\mathbf{J},\mathbf{h}})^2} \tag{3.14}$$

where $\mu^{\Lambda,\mathbf{J},\mathbf{h}} = \mu^{\Lambda,\mathbf{h}}$ and $\hat{Z}^{\Lambda,\mathbf{J},\mathbf{h}} = \sum_{\sigma \in S^\Lambda} \exp - H_{\Lambda,\mathbf{h}}(\sigma)$. Also, it easy to see that

$$\mu^{\Lambda,\mathbf{J},\mathbf{h}}(\sigma_x; \sigma_y) = \tfrac{1}{2}\mu^{\Lambda,\mathbf{J},\mathbf{h}} \otimes \mu^{\Lambda,\mathbf{J},\mathbf{h}}(\sigma_x^- \sigma_y^-) \tag{3.15}$$

By (3.14) and (3.15),

$$\begin{aligned}
&2(\hat{Z}^{\Lambda,\mathbf{J},\mathbf{h}})^2 \mu^{\Lambda,\mathbf{J},\mathbf{h}}(\sigma_x; \sigma_y)\\
&\quad= \sum_{\sigma^1,\sigma^2} \sigma_x^- \sigma_y^- \exp -2\{H_{\Lambda,\mathbf{h}}(\sigma^+) + H_{\Lambda,0}(\sigma^-)\}\\
&\quad= \sum_{(\sigma^+,\sigma^-) \in T} \sigma_x^- \sigma_y^- \exp -2\{H_{\Lambda,\mathbf{h}}(\sigma^+) + H_{\Lambda,0}(\sigma^-)\}\\
&\sum_{\Gamma \subset \Lambda} \sum_{\substack{(\sigma^+,\sigma^-):\, \sigma_x^- = \pm 1 \text{ on } \Gamma,\\ \sigma_x^+ = \pm 1 \text{ on } \Lambda\setminus\Gamma}} \sigma_x^- \sigma_y^- \exp -2\{H_{\Lambda,\mathbf{h}}(\sigma^+) + H_{\Gamma,0}(\sigma^-)\}
\end{aligned} \tag{3.16}$$

In the fourth line, we have decomposed the summation over $(\sigma^+, \sigma^-) \in T$ according to the "support" Γ of σ^-. The following trivial observation has also been used:

$$H_{\Lambda,0}(\sigma^-) = H_{\Gamma,0}(\sigma^-) \qquad \text{if} \quad \sigma^- \equiv 0 \quad \text{on} \quad \Lambda\setminus\Gamma$$

By (3.15) and Lemma 3.3, we have that

$$\begin{aligned}
\sum_{\sigma^- \in \{-1,+1\}^\Gamma} \sigma_x^- \sigma_y^- \exp -2H_{\Gamma,0}(\sigma^-) &= 2\hat{Z}^{\Gamma,2\mathbf{J},0} \mu^{\Gamma,2\mathbf{J},0}(\sigma_x \sigma_y)\\
&\leqslant 2C_4 \hat{Z}^{\Gamma,2\mathbf{J},0} \exp - \frac{|x-y|}{C_4}
\end{aligned}$$

Plugging this into (3.16), we end up with

$$(3.16) \leqslant 2C_4 \sum_{\Gamma \subset \Lambda} \hat{Z}^{\Gamma,2\mathbf{J},0} \hat{Z}^{\Lambda\setminus\Gamma,2\mathbf{J},2\mathbf{h}} \exp - \frac{|x-y|}{C_4} \tag{3.17}$$

Now, by repeating similar computations, we have

$$(\hat{Z}^{\Lambda,\mathbf{J},\mathbf{h}})^2 = \sum_{\Gamma \subset \Lambda} \hat{Z}^{\Gamma,2\mathbf{J},0} \hat{Z}^{\Lambda\setminus\Gamma,2\mathbf{J},2\mathbf{h}} \tag{3.18}$$

Putting (3.16)–(3.18) together, we conclude (3.12).

Step 2. We will prove (2.1). We first assume that $f(\sigma)=\sigma_z$ $(z\in\Lambda)$. In this case, we obtain

$$\sup_{\omega\in S^G}|\mu^{\Lambda,\omega^y}(\sigma_z)-\mu^{\Lambda,\omega}(\sigma_z)|=\int_0^1\frac{d}{d\theta}\mu^{\Lambda,\mathbf{h}(\theta)}(\sigma_z)\,d\theta$$

$$=-2\int_0^1\sum_{v\in\Lambda:\,|v-y|=1}J_{\{y,v\}}\,\omega_v\mu^{\Lambda,\mathbf{h}(\theta)}(\sigma_z;\sigma_v)\tag{3.19}$$

where we have defined the mgnetic field $\mathbf{h}(\theta)$ by

$$h_x(\theta)=h_x+\sum_{w\notin\Lambda:\,|w-x|=1}J_{\{x,w\}}\{\theta\omega_w^y+(1-\theta)\,\omega_w\}$$

But we have by (3.12) that

$$\mu^{\Lambda,\mathbf{h}(\theta)}(\sigma_z;\sigma_v)\leqslant C_4\exp-\frac{|v-z|}{C_4}$$

$$\leqslant C_4\exp-\frac{|y-z|-1}{C_4}$$

Plugging this into (3.19), we obtain (2.1) for the case $f(\sigma)=\sigma_z$. The general case follows from ref. 1, Lemma(2.1). QED

ACKNOWLEDGMENTS

We thank T. Kumagai for various comments which encouraged the authors to improve a previous version of our results. We also thank T. Funaki, T. Chiyonobu, and M. Sugiura for discussions.

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Communicated by J. L. Lebowitz